Connectivity of Friends-and-strangers Graphs

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Overview

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Graph Definition

Definition

A graph G is a pair $(V, E) = (V(G), E(G))$ where V is a set of elements called vertices and E is a set of tuples of vertices in V. In G, there exists an edge between two vertices v_1 and v_2 if $(v_1, v_2) \in E$.

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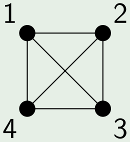

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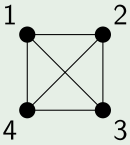

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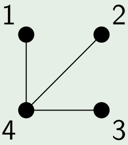

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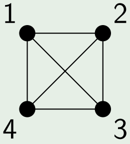

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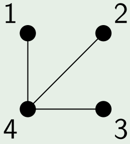

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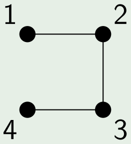

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- C_n — the cycle graph.

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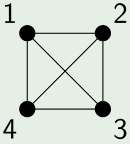

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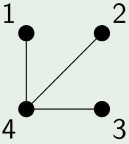

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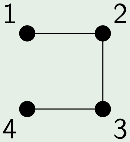

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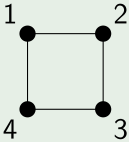

Induced Subgraphs and Minimum Degree

Definition

For some subset V_0 of $V(G)$, the induced subgraph denoted by $G|_{V_0}$ is the graph H where $V(H) = V_0$ and $E(H) = \{(i,j) : i, j \in V_0, (i,j) \in E(G)\}$.

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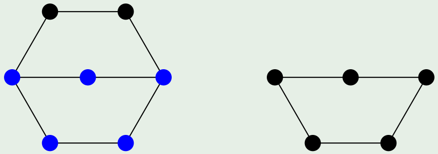

Graph and the subgraph induced by blue vertices.

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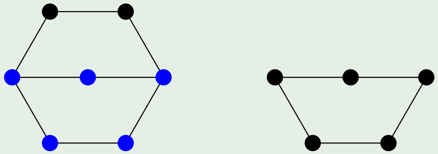

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Definition

The minimum degree of a graph G is smallest degree of a vertex in G. It is denoted $\deg(G)$.

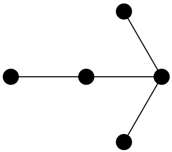
Overview

1. Preliminaries
2. Connectivity
3. Friends-and-strangers Graphs
4. Results

Classical Connectivity

Definition

A graph G is connected if for any two vertices $u, v \in V(G)$, there exists a path between u and v.



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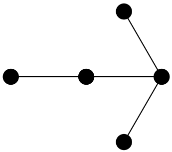

A connected graph

This notion can be extended.

k-connectivity

Definition

A graph G is k-connected if $|V(G)| > k$ and when any $k-1$ vertices are removed, the graph induced by the remaining vertices is connected.

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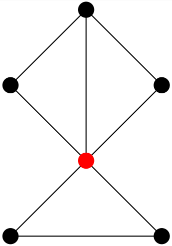

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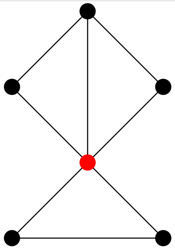

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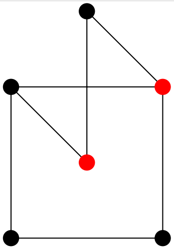

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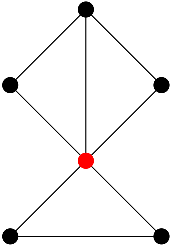

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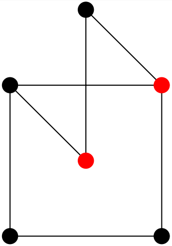

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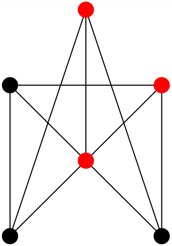
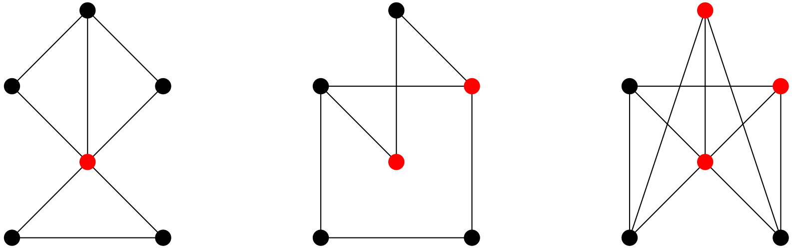


Connectivity of 3.

Properties of k-connectivity



Connectivity of 1. Connectivity of 2. Connectivity of 3.

Proposition

If G is a graph, then $\deg(G)$ *is greater than or equal to the connectivity of* G.

Definition

A graph G is maximally connected if the connectivity of G equals $\deg(G)$.

Overview

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$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

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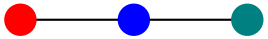
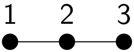
Example

Let $\sigma = 12345 \to 14253$. Then $\sigma \circ (1, 2) = 12345 \to 41253$ while $(1, 2) \circ \sigma = 12345 \to 24153$.

Friendly Swaps

Let $X =$  and let $Y =$ 

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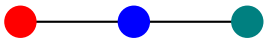
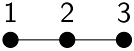
Let $X =$  and let $Y =$ 

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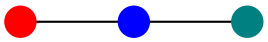
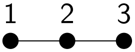
Let $X =$  and let $Y =$ 

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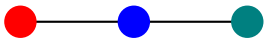
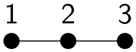
Let $X =$  and let $Y =$ 

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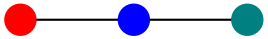
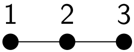
Let $X =$  and let $Y =$ 

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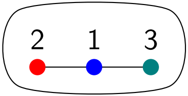

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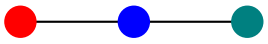
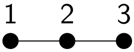
Let $X =$  and let $Y =$ 

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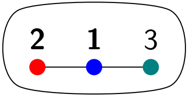

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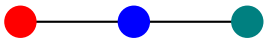
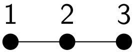
Let $X =$  and let $Y =$ 

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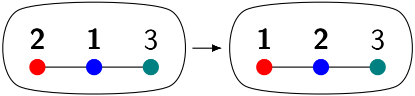

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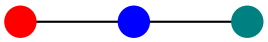
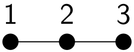
Let $X =$  and let $Y =$ 

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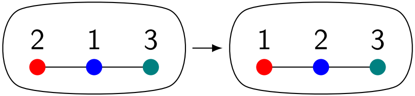

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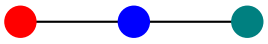
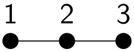
Let $X =$  and let $Y =$ 

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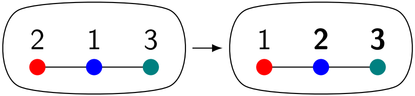

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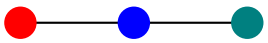
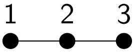
Let $X =$  and let $Y =$ 

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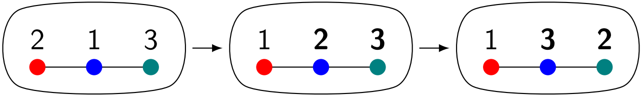

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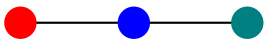
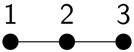
Let $X =$  and let $Y =$ 

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- In X, vertices are position and edges mark adjacent positions. In Y, vertices are people and edges mark friendships.
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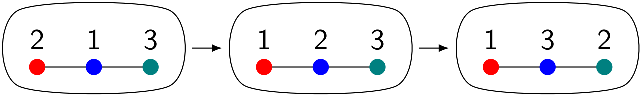

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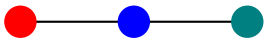
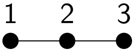
Let $X =$  and let $Y =$ 

Definition

The friends-and-strangers graph of X and Y both on n vertices denoted $\mathrm{FS}(X, Y)$ is the graph whose vertices are the placements of people on the positions where two placements are connected by an edge if it is possible to perform a friendly-swap on one placement to get to the other.

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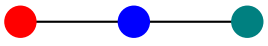
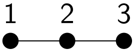
Let $X =$  and let $Y =$ 

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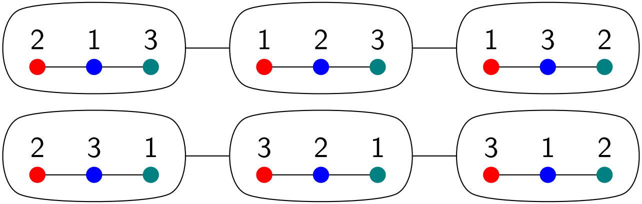

The graph $\mathrm{FS}(X, Y)$.

Formal Friends-and-strangers Graphs

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Given X and Y, the friends-and-strangers graph $\mathsf{FS}(X, Y)$ is defined as follows. Its vertex set is the set of bijections $\sigma : V(X) \to V(Y)$, and (σ, τ) is an edge if and only if $\sigma = \tau \circ (i, j)$ for some $i, j \in V(X)$ with $(i, j) \in E(X)$ and $(\sigma(i), \sigma(j)) \in E(Y)$. In this case, we say σ and τ differ by an (X, Y)-friendly swap.

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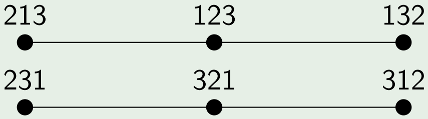
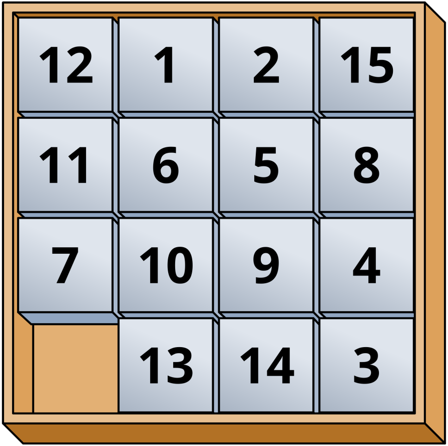


The graph $\mathrm{FS}(P_3, P_3)$.

The 15-puzzle

Consider the 15-puzzle where tiles numbered 1 to 15 are placed in a 4×4 grid. Tiles can then slide around but they can never overlap.



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Examples of slide moves in the 15-puzzle.

We can think about this puzzle as the 16 squares of the grid being positions and the tiles being people. The square without a tile can be thought of as a person who is friends with everyone.

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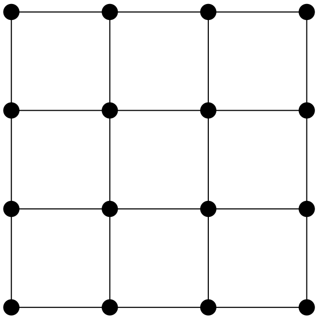

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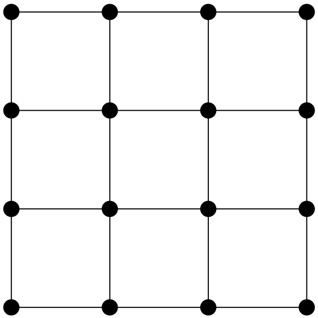

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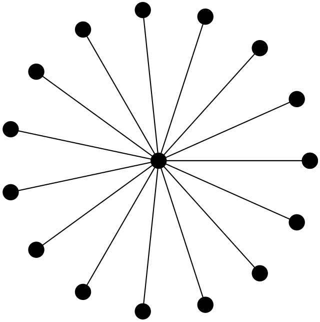


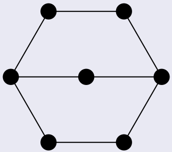
The graph $Y = S_{16}$.

Wilson's Theorem

Theorem (Wilson, 1974)

Suppose that X is a graph on $n \geq 3$. If it satisfies the following properties, then $\mathrm{FS}(X, S_n)$ is connected.

- *X is biconnected,*
- *X is not bipartite,*
- *X is not isomorphic to C_n for $n \geq 4$,*
- *X is not isomorphic to the graph θ_0 on 7 shown below.*



The graph θ_0.

Minimum Degree Conditions

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Theorem (Bangachev, 2022)

Suppose that X and Y are two graphs on $n \geq 6$ vertices satisfying:

- $\deg(X), \deg(Y) > n/2$,
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Then $\mathrm{FS}(X, Y)$ is connected.

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Let X and Y be graphs independently chosen from $G(n, p)$, i.e., they are graphs on n vertices and edges occur with probability p.

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Overview

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Conjecture (K.–Li, 2024++)

For all X and Y where $\mathrm{FS}(X, Y)$ is connected, $\mathrm{FS}(X, Y)$ is maximally connected.

Minimum Degree

The overall theme for the rest of the results is that if $\mathsf{FS}(X, Y)$ is connected for X and Y on n vertices, then it is close to k-connected for sufficiently large n.

Theorem (K.–Li, 2024++)

Let X and Y be two graphs on n vertices satisfying

- $\deg(X), \deg(Y) > n/2$,
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For fixed k, the bound conditions are $\sim n$ and $\sim 3n$ which are the same as the bounds for classical 1-connectivity proven by Bangachev.

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Question (Milojević 2022)

Let $p(n) = n^{-1/2+o(1)}$ *and let* X *and* Y *be random graphs in* $G(n, p)$. *For which values of* k *is the graph* $\mathsf{FS}(X, Y)$ k*-connected with high probability?*

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As a result, if $p(n) = n^{-1/2+o(1)}$, $\mathrm{FS}(X,Y)$ is k-connected with high probability.

Acknowledgements

I would like to thank my mentor, Rupert Li, for introducing me to friends-and-strangers graphs and helping me with the research process.

I also thank Dr. Tanya Khovanova of the Department of Mathematics at the Massacheusetts Institute of Technology (MIT) for her advice. I thank Dr. Slava Gerovith and Prof. Pavel Etingof of the Department of Mathematics at MIT for creating and operating the PRIMES-USA program which has given this opportunity.

References I

N. Alon, C. Defant, and N. Kravitz, *Typical and extremal aspects of friends-and-strangers graphs*, J. Combin. Theory Ser. B **158** (2023), 3–42. MR 4513816 10.1016/j.jctb.2022.03.001

K. Bangachev, *On the asymmetric generalizations of two extremal questions on friends-and-strangers graphs*, European J. Combin. **104** (2022), Paper No. 103529, 26. MR 4400016 10.1016/j.ejc.2022.103529

C. Defant and N. Kravitz, *Friends and strangers walking on graphs*, Comb. Theory **1** (2021), Paper No. 6, 34. MR 4396211 10.5070/C61055363

R. Diestel, *Graph theory*, fifth ed., Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2018. MR 3822066

References II

- C. D. Godsil, *Connectivity of minimal Cayley graphs*, Arch. Math. (Basel) **37** (1981), no. 5, 473–476. MR 643291 10.1007/BF01234384

- F. Göring, *Short proof of Menger's theorem*, Discrete Math. **219** (2000), no. 1-3, 295–296. MR 1761733 10.1016/S0012-365X(00)00088-1

- R. Jeong, *On the diameters of friends-and-strangers graphs*, Comb. Theory **4** (2024), no. 2, Paper No. 2. 10.5070/C64264229

- K. Menger, *Zur allgemeinen kurventheorie*, Fund. Math. **10** (1927), no. 1, 96–115.

- A. Milojević, *Connectivity of old and new models of friends-and-strangers graphs*, Adv. in Appl. Math. **155** (2024), Paper No. 102668, 53. MR 4689232 10.1016/j.aam.2023.102668

References III

C. Reidys, *Acyclic orientations of random graphs*, Advances in Applied Mathematics **21** (1998), no. 2, 181–192.

R. P. Stanley, *An equivalence relation on the symmetric group and multiplicity-free flag h-vectors*, Journal of Combinatorics **3** (2013), no. 3, 277–298.

L. Wang and Y. Chen, *Connectivity of friends-and-strangers graphs on random pairs*, Discrete Math. **346** (2023), no. 3, Paper No. 113266, 10. MR 4513695 10.1016/j.disc.2022.113266

M. E. Watkins, *Connectivity of transitive graphs*, J. Combinatorial Theory **8** (1970), 23–29. MR 266804

R. M. Wilson, *Graph puzzles, homotopy, and the alternating group*, J. Combinatorial Theory Ser. B **16** (1974), 86–96. MR 332555 10.1016/0095-8956(74)90098-7